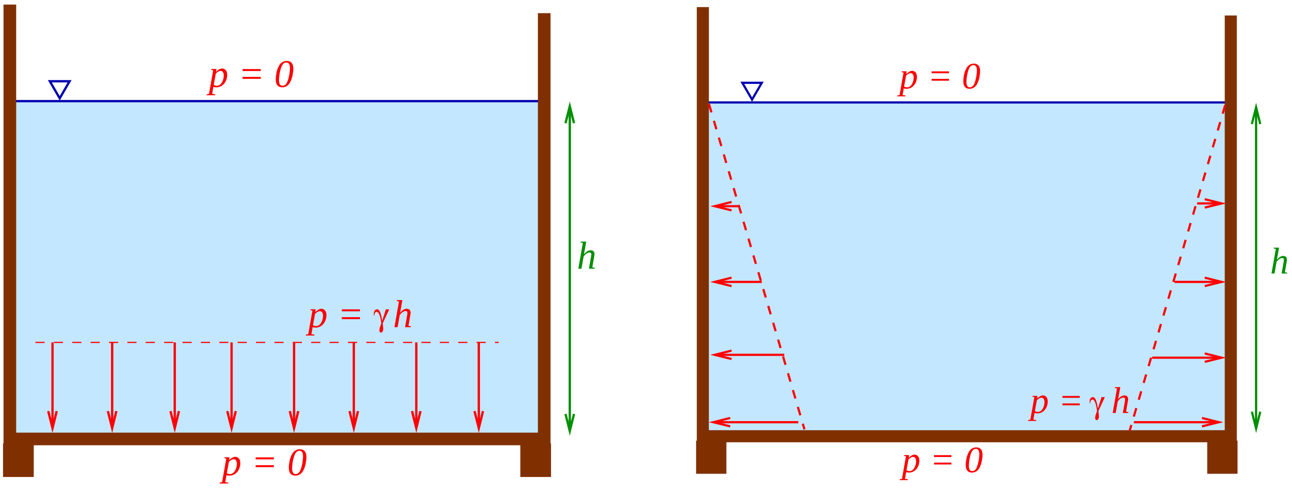
Hydrostatic Forces and Buoyancy

The Joola after capsizing (Senegal 2003)

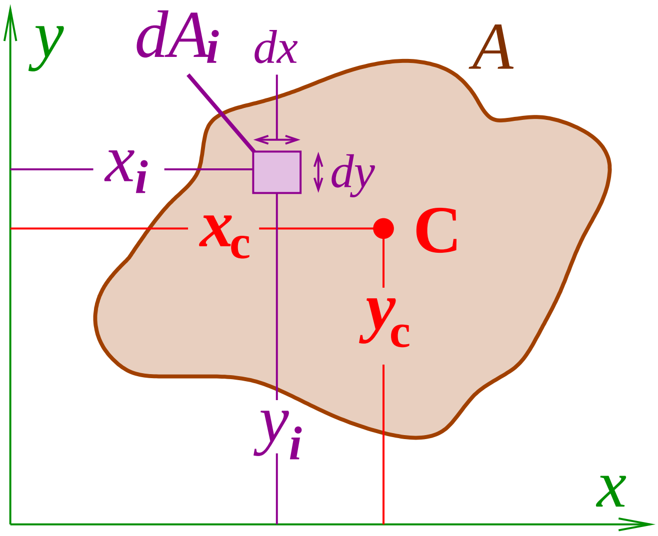
Hydrostatics Force



- The pressure on the bottom is uniform so the resultant force acts through the centroid.
- The pressure of the sides increases with decreasing depth. The force will not act through the centroid of the surface.
- The centroid is the geometric mean position center of the surface. The line of action (center of pressure) weights the area integral by the force applied through that area.

Definition of the centroid

The centroid gives a definition of the mean position of an area (volume). It is closely related to the center of mass of a body.



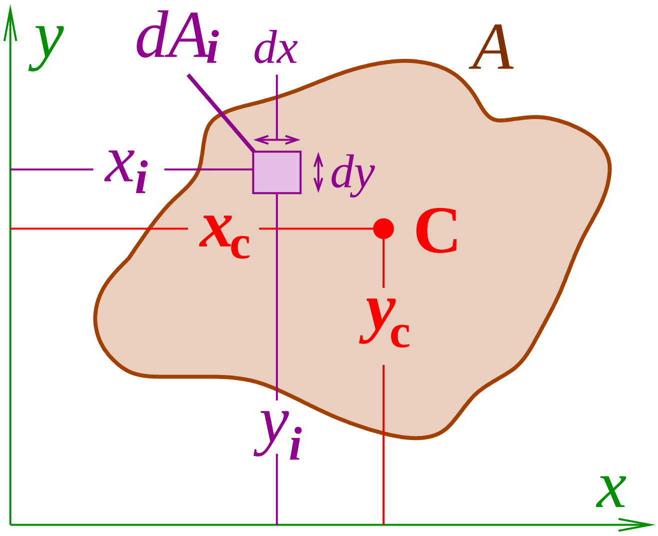
One adds up position of x for all the little pieces dA_i of the Area, A to get average x position, x_c . The x and y -coordinates of the centroid are evaluated mathematically as

$$x_c = \frac{\sum_i x_i \, dA_i}{\sum_i dA_i} = \frac{\iint_A x \, dx \, dy}{A}$$

$$y_c = \frac{\sum_i y_i \, dA_i}{\sum_i dA_i} = \frac{\iint_A y \, dx \, dy}{A}$$

First moment of Area

The 1st moments of areas are the average displacement of an area about an axis of rotation. They are closely related to the centroid.



The first moment of area about the y -axis is

$$Q_y = \sum_i x_i dA_i = \iint_A x\ dx\ dy$$

$$Q_y = \iint_A (x - x_c)\ dx\ dy + x_c \iint_A dx\ dy$$

$$Q_y = 0 + x_c \iint_A dx\ dy = x_c A$$

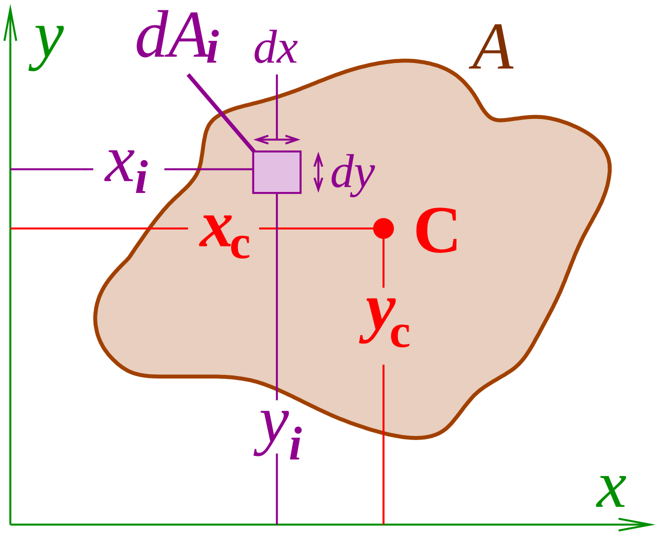
The first moment of area about the x -axis is

$$Q_x = \iint_A y\ dx\ dy = y_c A$$

The first moments of area have units of m^3 .

Second moment of Area

The 2nd moments of areas are the average *(displacement)*^{2} of an area about an axis of rotation. Has units of m^4



The second moment of area about the x -axis is

$$I_x = \sum_i y_i^2 dA_i = \iint_A y^2 dx\ dy$$

It is sometimes called the *moment of inertia of the area*. The second moment of inertia is always positive since $y^2 > 0$.

The second moment of area about the y -axis is

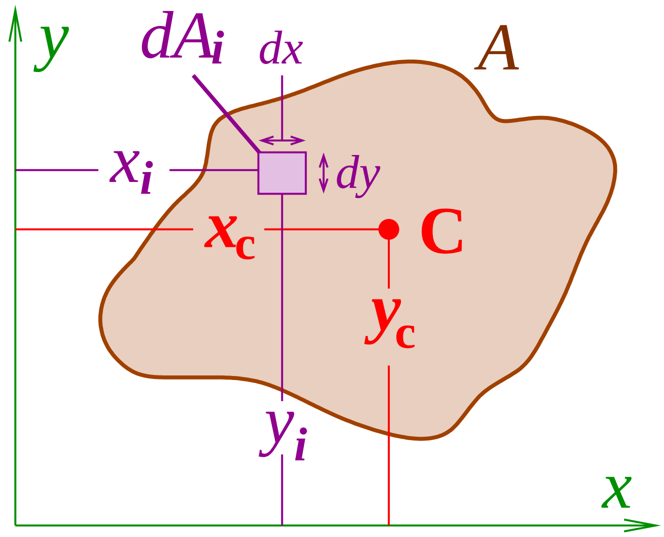
$$I_y = \sum_i x_i^2 dA_i = \iint_A x^2 dx\ dy$$

The product of inertia about an xy coordinate axes

$$I_{xy} = \sum_i x_i y_i dA_i = \iint_A xy dx\ dy$$

Parallel axis theorem

Working out the second moments would be troublesome as the axes of rotations moved but for the parallel axes theorem. The moments of many objects through their centroids are known.



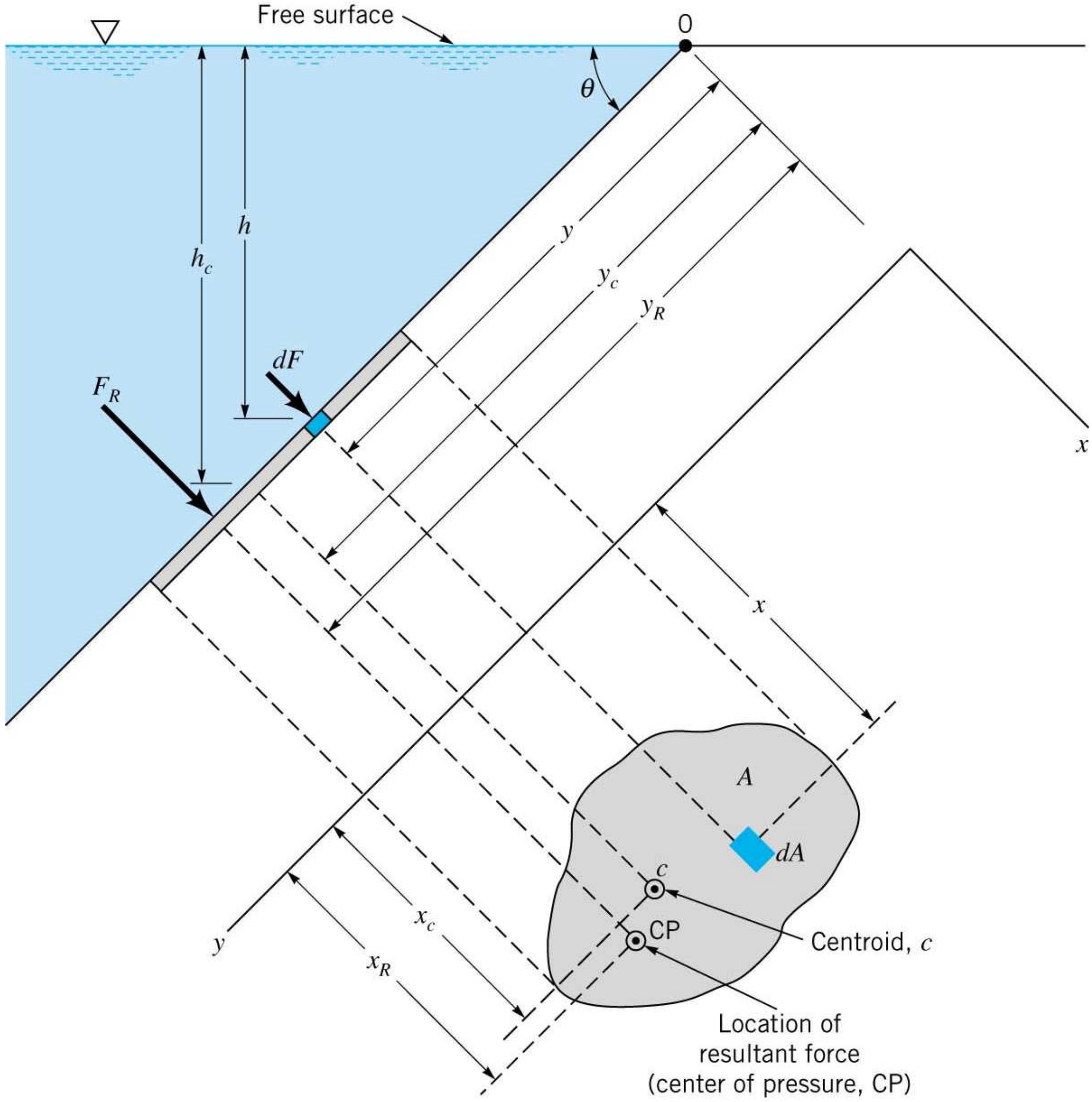
The second moment of areas are

$$
\begin{aligned}
I_x &= I_{xc} + y_c^2 A \\
I_y &= I_{yc} + x_c^2 A \\
I_{xy} &= I_{xyc} + x_c y_c A
\end{aligned}
$$

One writes down second moment through centroid, then determines distance of centroid to axis of rotation, and finally apply the parallel axis theorem.

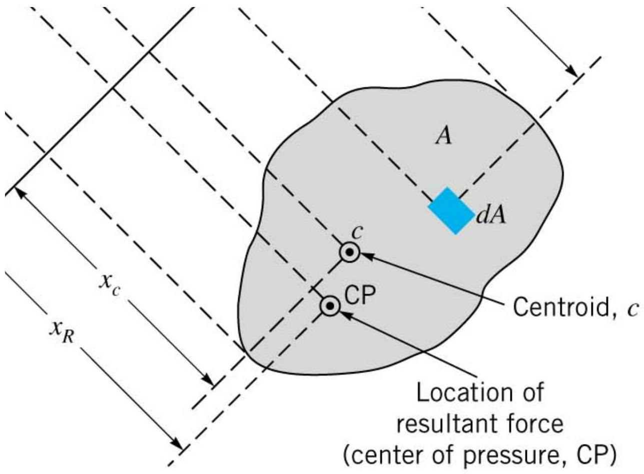
Submerged inclined plane

Want to work out forces on inclined surface.



- The x -axis points out of page.
- The distance down incline is y . Depth is h .
- $p = \gamma h$ (gauge pressure).

Inclined plane



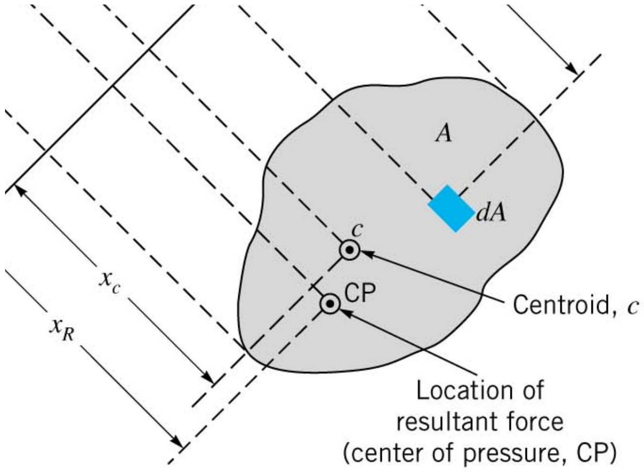
To determine net force, need to add up all contributions over each small piece of the area.

$$
\begin{aligned}
dF &= p\, dA = \gamma h\, dA \quad \text{one piece of A} \\
F_R &= \iint_A \gamma h\, dA \quad \text{adding up} \\
F_R &= \iint_A \gamma y \sin\theta\, dA \quad h = y \sin\theta \\
F_R &= \gamma \sin\theta \iint_A y\, dA \quad \text{if } \gamma \text{ and } \theta \text{ constant}
\end{aligned}
$$

Now the integral over y is the first moment of Area,

$$\iint_A y\, dA = y_c A$$

Inclined plane

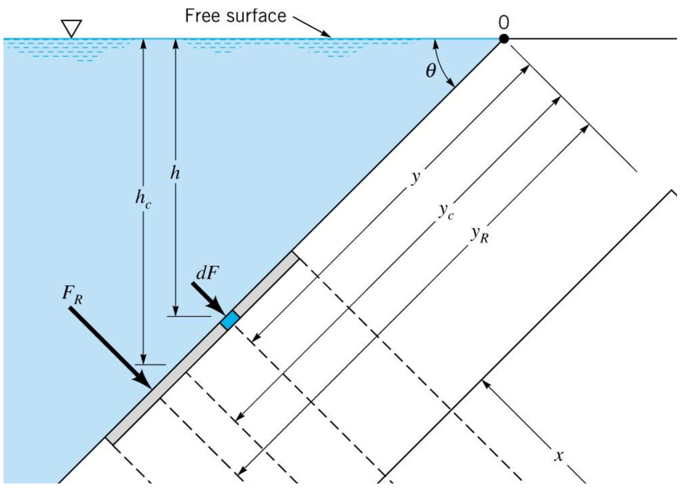


$$\iint_A y\, dA = y_c A$$

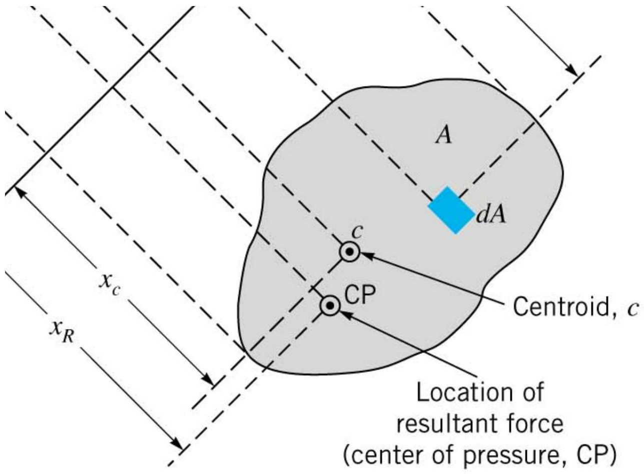
Since $F = \gamma \sin\theta \iint_A y\, dA$

$$F_R = \gamma A y_c \sin\theta = \gamma h_c A \qquad h_c = y_c \sin\theta$$

The net force on the plane depends on the depth of the plate centroid below the surface. The net force is the area multiplied by the pressure at the centroid.



Center of pressure



To determine the center of pressure, add up all contributions to the force over each small piece of the area, but multiplied by y .

The center of pressure is essentially a weighted average.

$$\langle y \rangle = \frac{\sum_i y_i \delta F_i}{\sum_i \delta F_i}$$

$$y_R = \frac{\iint_A y \; dF}{\iint_A \; dF}$$

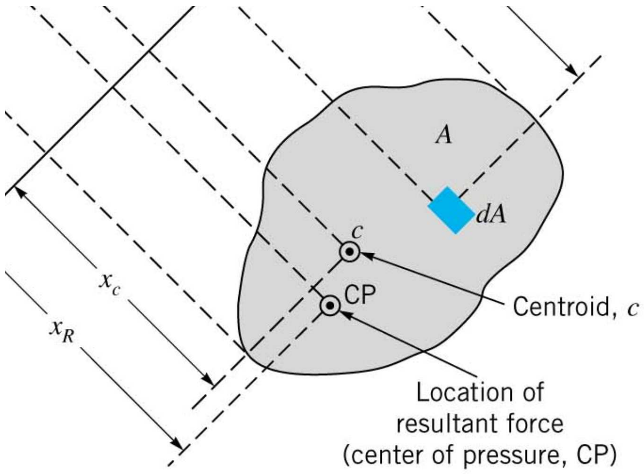
$$y_R = \frac{\iint_A yp \; dA}{\iint_A p \; dA}$$

$$y_R = \frac{\iint_A y \gamma y \sin\theta \; dA}{\iint \gamma \sin\theta y \; dA}$$

$$y_R = \frac{\iint_A y^2 \; dA}{\iint y \; dA} = \frac{\iint_A y^2 dA}{y_c A}$$

Center of pressure

$$y_R = \frac{\iint_A y^2 dA}{y_c A}$$



The y position of the force is the 2nd moment of the area with respect to the x axis. This is essentially the moment of inertia about the x -axis.

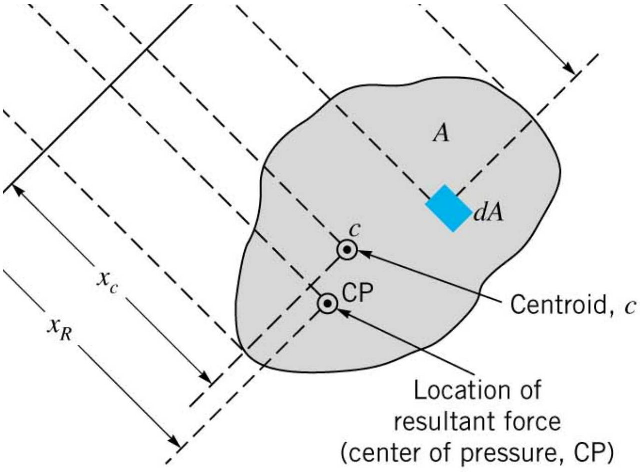
$$y_R = \frac{I_x}{A y_c}$$

Parallel axis theorem $I_x = I_{xc} + A y_c^2$.

$$y_R = \frac{I_{xc} + A y_c^2}{A y_c} = \frac{I_{xc}}{A y_c} + y_c$$

The resultant force F_R always passes below the centroid since $y_R > y_c$.

Center of Pressure, x



$$x_R = \frac{\iint_A xy dA}{y_c A}$$

The mean x -position of the force can be determined by similar technique. This is just the product of inertia for the coordinate system.

$$x_R = \frac{\iint_A xy dA}{y_c A} = \frac{I_{xy}}{y_c A}$$

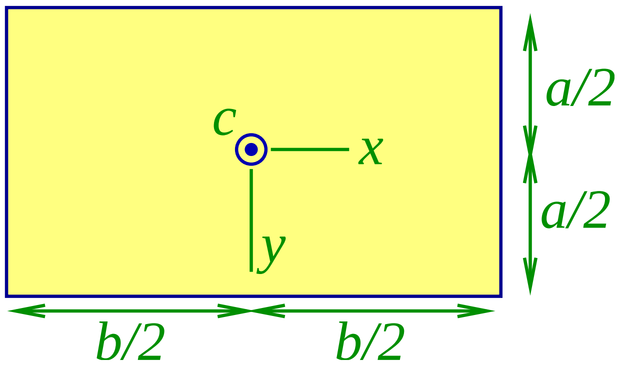
$$x_R = \frac{I_{xyc} + x_c y_c A}{y_c A}$$

$$x_R = \frac{I_{xyc}}{y_c A} + x_c$$

Geometric properties for shapes

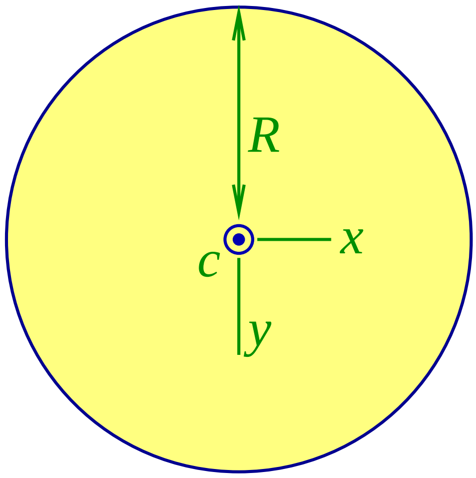
Rectangle

$$A = ba$$
$$I_{xc} = ba^3/12$$
$$I_{yc} = ab^3/12$$
$$I_{xyc} = 0$$



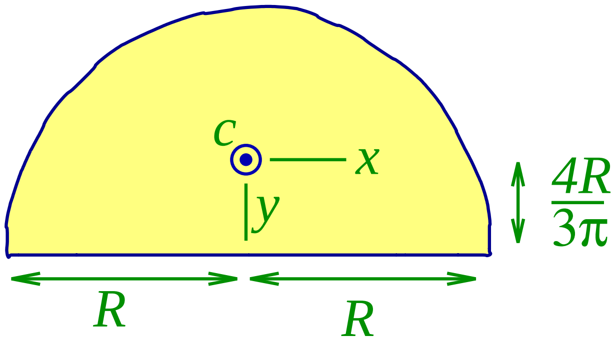
Circle

$$A = \pi R^2$$
$$I_{xc} = I_{yc} = \pi R^4/4$$
$$I_{xyc} = 0$$



Half-circle

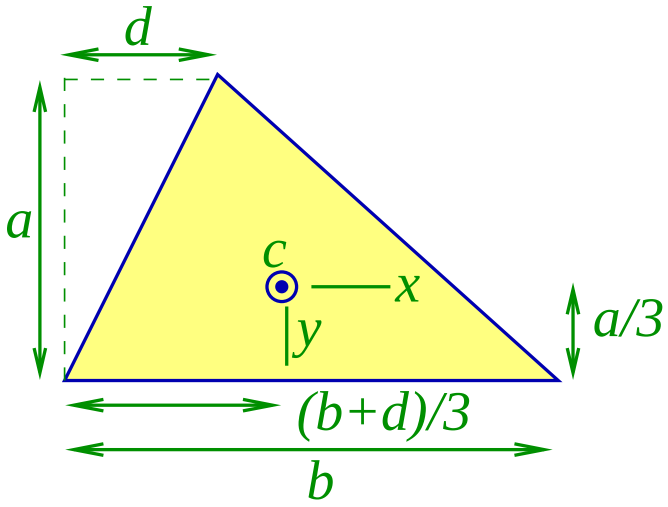
$$A = \pi R^2/2$$
$$I_{xc} = 0.1098R^4$$
$$I_{yc} = 0.3927R^4$$
$$I_{xyc} = 0$$



More shapes

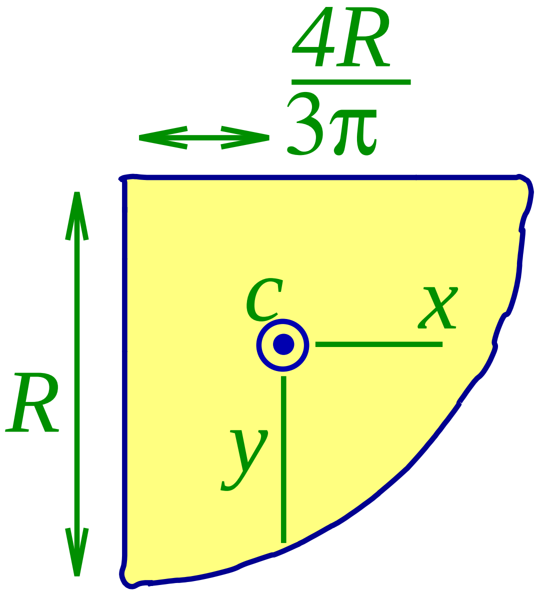
Triangle

$$A = ba/2$$
$$I_{xc} = ba^3/36$$
$$I_{xyc} = ba^2(b-2d)/72$$



Quarter Circle

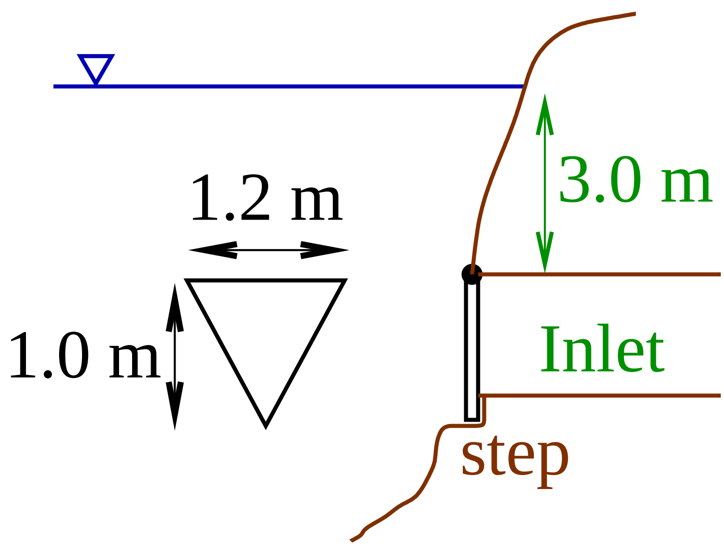
$$A = \pi R^2/4$$
$$I_{xc} = I_{yc} = 0.05488R^4$$
$$I_{xyc} = -0.01647R^4$$



I_{xyc} is only non-zero if the shape does not have a bilateral symmetry.

Worked example

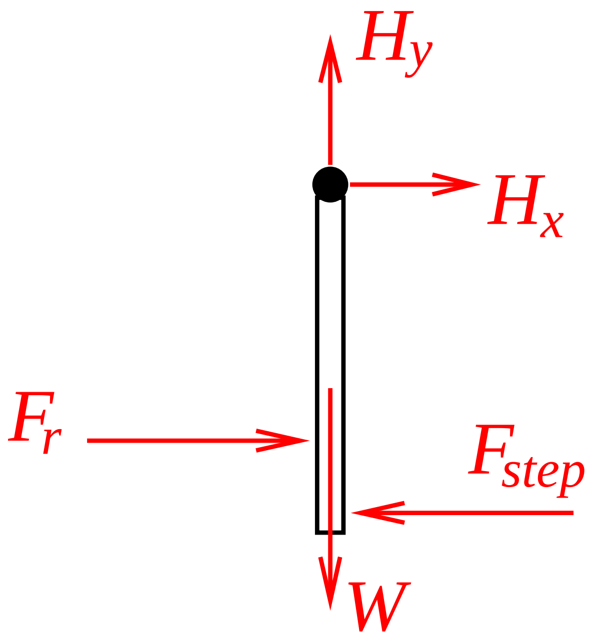
Determine the resultant force on the plate and the reaction at the step.



$\rho = 1000\ \mathrm{kg/m^3}$

Draw FBD for plate.

The weight force can be ignored during the calculation and $W = H_y$



Pipe-inlet

Determine resultant force

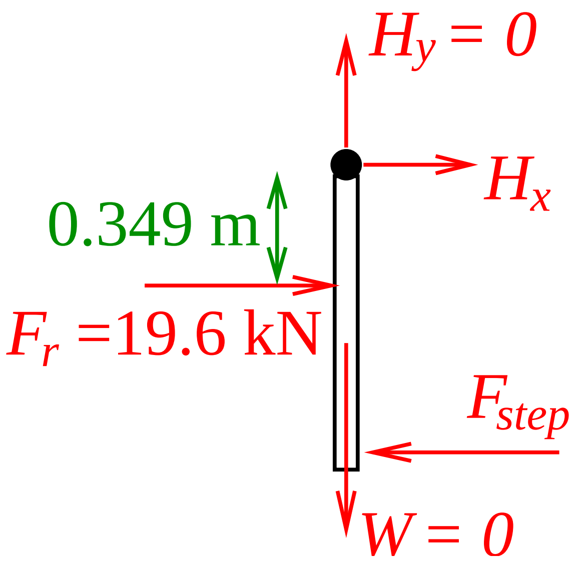
$$
\begin{aligned}
F_r &= \rho g h_c A \\
A &= \frac{1}{2}ab = \frac{1}{2}1.2 \times 1.0 = 0.60 \text{ m}^2 \\
h_c &= 3.0 + \frac{a}{3} = 3.0 + \frac{1}{3}1.0 = 3.333 \text{ m} \\
F_r &= 1000 \times 9.810 \times 0.600 \times (3 + 0.333) = 19.6 \text{ kN} \\
I_{xc} &= \frac{1.2 \times 1.0^3}{36} = 0.0333 \text{ m}^4
\end{aligned}
$$

Now for center of pressure

$$
\begin{aligned}
y_r &= \frac{I_{xc}}{A y_c} + y_c \\
y_r &= \frac{0.0333}{0.60 \times 3.333} + 3.333 \\
y_r &= 0.016 + 3.333 \\
y_r &= 3.349 \text{ m}
\end{aligned}
$$

Pipe-inlet: Step reaction

Net torque about hinge must be zero.



$$F_{step} \times 1.0 = F_r(0.349)$$

$$F_{step} = 19.6 \times 10^3(0.349)$$

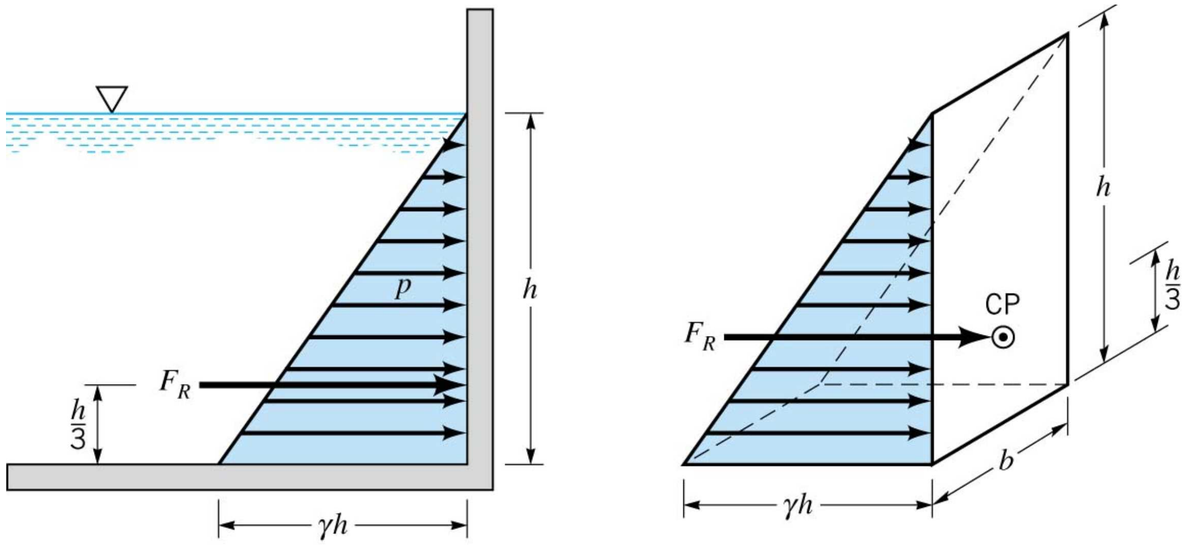
$$F_{step} = 6.85 \text{ kN}$$

The force on the hinge determined from $F_r - F_{step} + H_x = 0$.

$$H_x = F_{step} - F_r = 6.85 - 19.6 = -12.7\text{kN}$$

The force on the hinge acts to the left (opposes F_r)

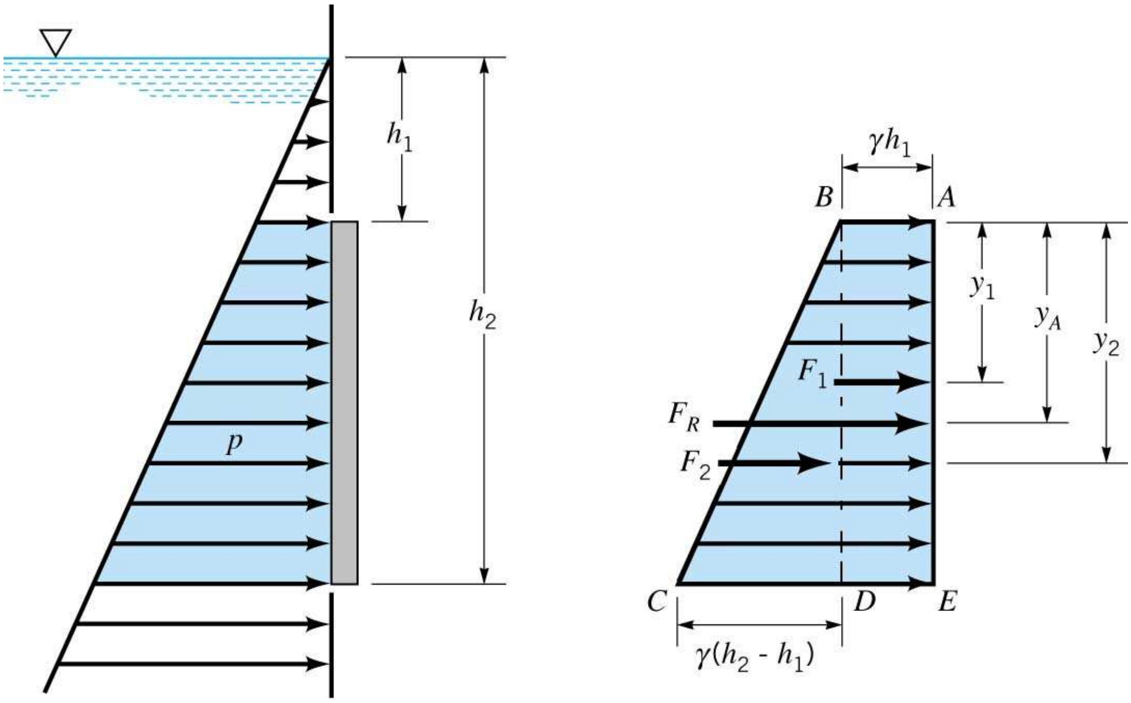
Pressure Prism



This is a intuitive recipe for determining the force on submerged surfaces. Useful for surfaces that are rectangular in shape.

- Gauge pressure is zero at top and γh at bottom.
- Pressure variation with h is linear.
- Average pressure $\langle p \rangle = \gamma h/2$
- Resultant force $F_r = \langle p \rangle A = \gamma(hA/2)$
- Volume of pressure prism $(= \gamma hA/2)$.
- The center of pressure passes through the centroid of the pressure prism.

Pressure Prism



The pressure prism can be regarded as arising from 2 parts. Let w be width of surface into page. Force due to rectangle ABDE .

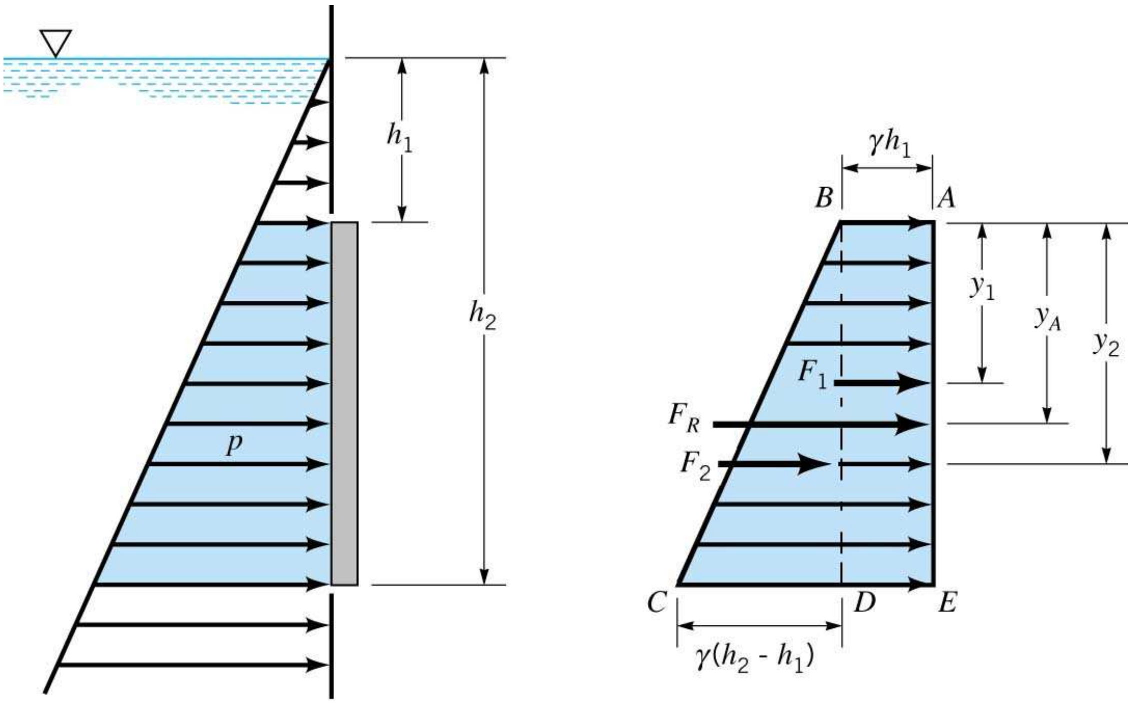
$$F_1 = (\gamma h_1)(h_2 - h_1)w$$

Force due to triangle BCD.

$$F_2 = \left(\frac{1}{2}\gamma(h_2 - h_1)\right)(h_2 - h_1)w$$

The resultant force is $F_R = F_1 + F_2$

Pressure Prism

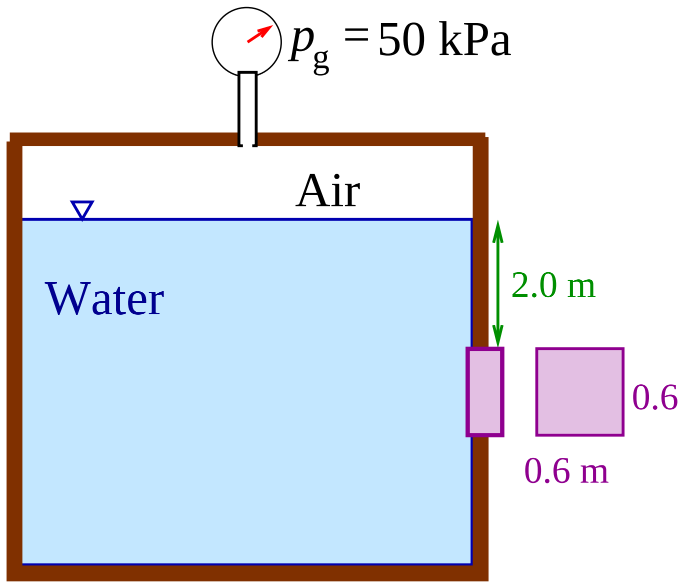


Determination of Center of Pressure done from moments of the forces.

$$F_R y_R = F_1 y_1 + F_2 y_2$$
$$\Rightarrow y_R = \frac{F_1 y_1 + F_2 y_2}{F_R}$$

Moment of rectangular part about AB level is $1/2$ distance apex to base, i.e. $y_1 = \frac{1}{2}(h_2 - h_1)$. Moment of triangular part is $2/3$ of distance from apex to base, i.e. $y_2 = \frac{2}{3}(h_2 - h_1)$.

The tank problem

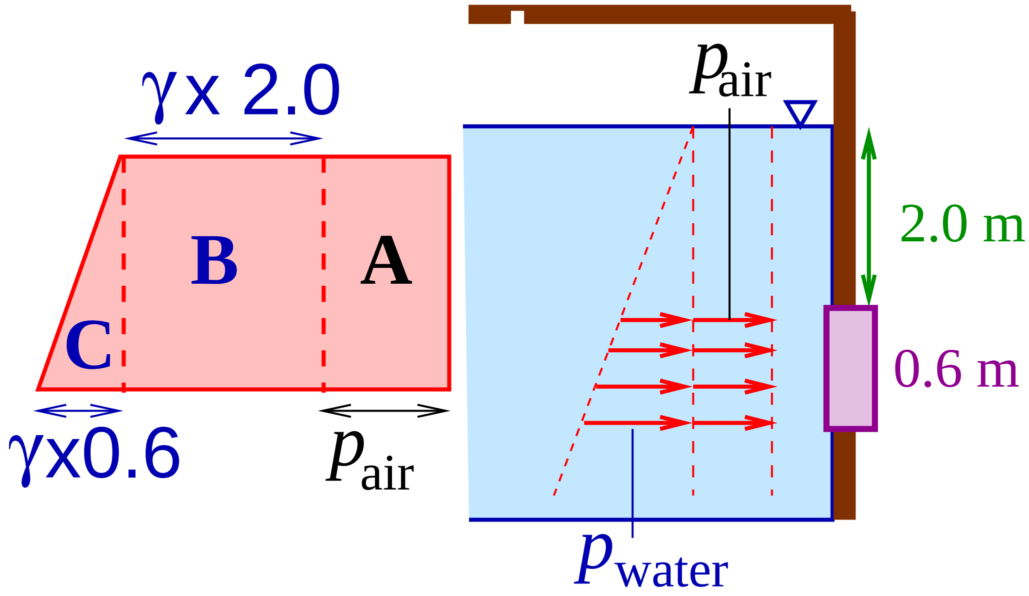


Want to determine the force on the cover plate.

Will also determine the center of pressure.

Useful hint: While the air pressure inside the tank needs to be taken into consideration, the impact of atmospheric pressure can be ignored (the tank pressure is a gauge pressure).

Tank Problem, pressure prism



The over-pressure due to air in the tank, 50 kPa is constant with depth. Pressures and forces due to (A+B) and C are

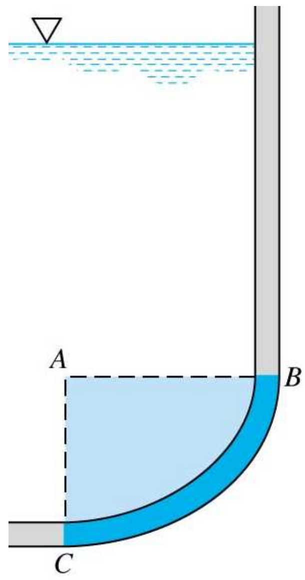
$$F_{A+B} = (50,000 + 2.0 \times 9,800) \times 0.6^2 = 25,060 \text{ N}$$
$$F_C = (\tfrac{1}{2}0.6 \times 9,800) \times 0.6^2 = 1060 \text{ N}$$

Total force is $F_R = 26,100$ N . The center of pressure is

$$y_R = \frac{25060 \times 0.3 + 1060 \times 0.400}{26100} = 0.304 \text{ m}$$

Force on curved surface

Curved surfaces occur in many structures, e.g. dams and cross sections of circular pipes.

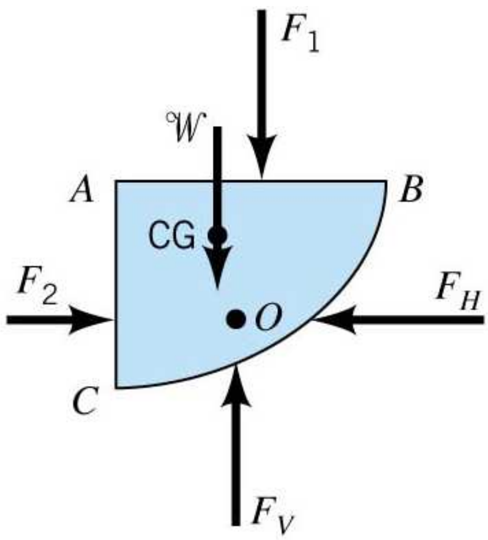


The loads on the surface are all due to pressure forces. Look at forces acting on wedge of water ABC .

Weight force W due to weight of wedge of water.

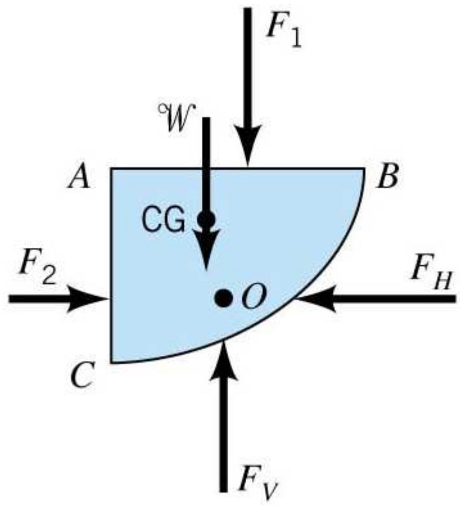
Pressure forces F_1, F_2 due to water above and from left.

Reaction Forces F_H, F_V due to wall of tank.



Curved surface: Free body diagram

The weight force W passes through the center of gravity of the wedge.



For static equilibrium,

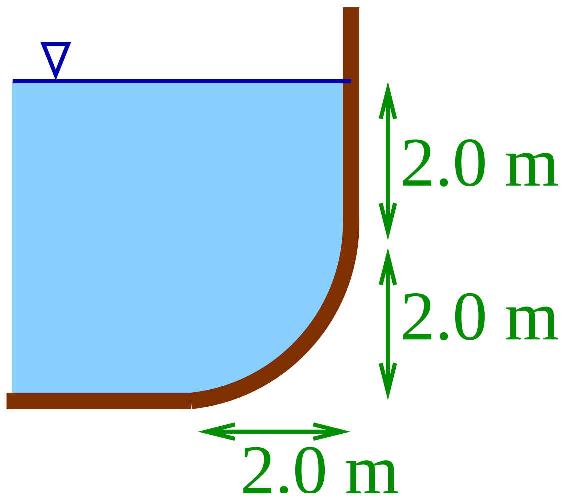
$$F_1 + W = F_V$$
$$F_2 = F_H$$

Also F_2 is co-linear with F_H and F_V is co-linear with the resultant of $F_1 + W$.

Curved Surface, example

Determine the resultant force on the curved part of the base and also determine its line of action.

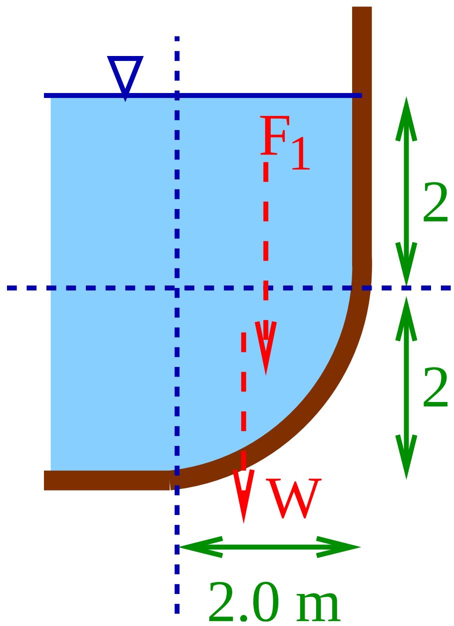
The bottom corner of the tank is a circle of radius 2.0 m .



The tank length (out of page) is 8.0 m .

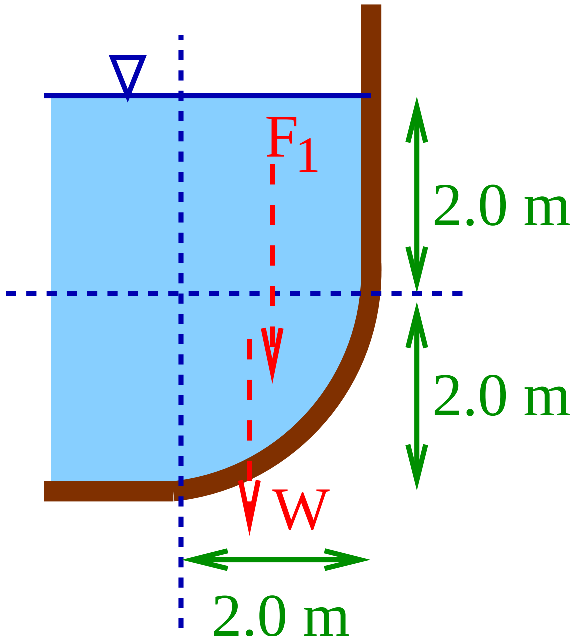
The centroid of the quarter circle wedge is

$$x_c = \frac{4R}{3\pi}$$
$$x_c = 0.84883 \text{ m}$$



It is 0.84883 m from the left boundary of quarter circle.

Curved Surface, Vertical



$$F_1 = \gamma 2.0(2.0 \times 8.0)$$
$$F_1 = 313.6 \text{ kN}$$

$$W = \gamma \frac{1}{4}\pi(2.0)^2 \times 8.0$$
$$W = 246.3 \text{ kN}$$

The total vertical force is $246.3 + 313.6 = 559.9$ kN

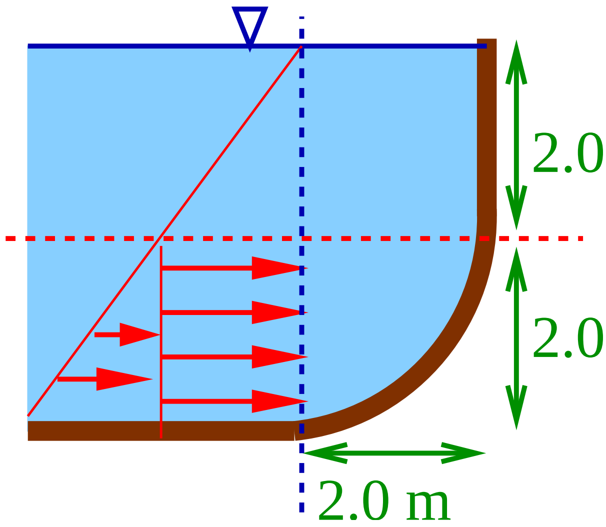
F_1 line of action 1.0 m from wall.

W line of action $2.0 - 0.84883 = 1.151$ m from wall.

Line of action for F_1 and W.

$$x_R = \frac{313.6 \times 1.0 + 246.3 \times 1.151}{559.9}$$
$$x_R = 1.066 \text{ m}$$

Curved Surface, Horizontal



Rectangle

$$F_{\Box} = \gamma 2.0(2.0 \times 8.0)$$
$$F_{\Box} = 313.6 \text{ kN}$$

$$F_{\triangle} = \gamma \frac{1}{2} 2.0(2.0 \times 8.0)$$
$$F_{\triangle} = 156.8 \text{ kN}$$

The net horizontal force is $313.6 + 156.8 = 470.4$ kN

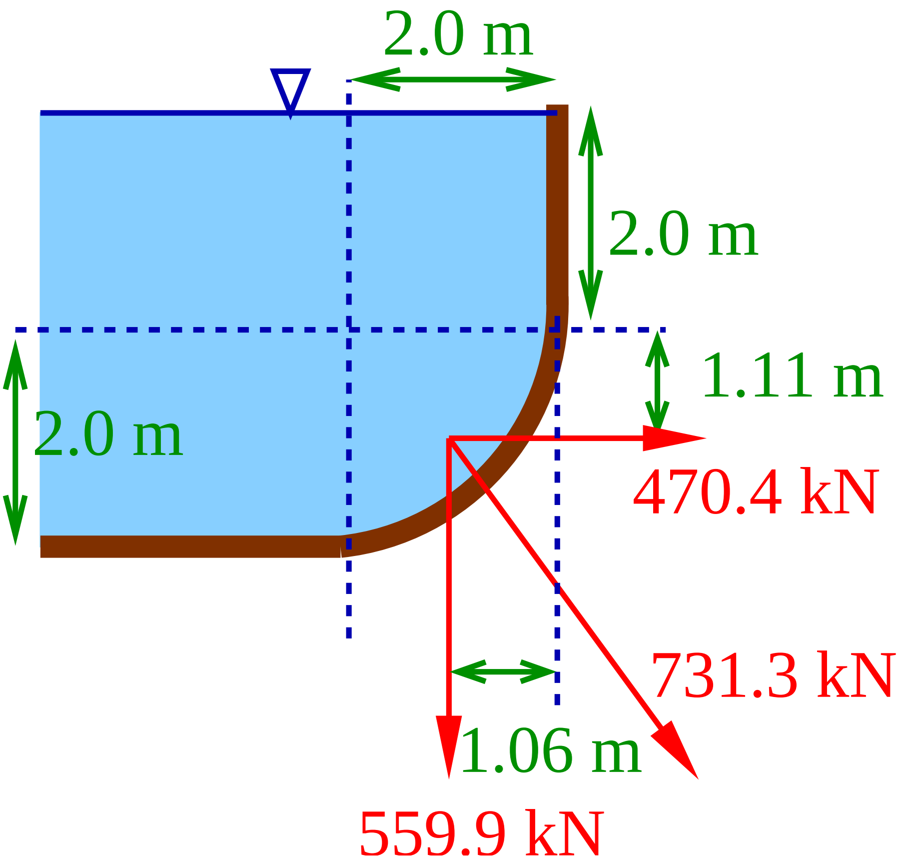
$F_{\Box}$ line of action 1.0 m below 2.0 m line.

$F_{\triangle}$ line of action 1.33 m below 2.0 m line.

Line of action for horizontal force.

$$y_R = \frac{313.6 \times 1.0 + 156.8 \times 1.333}{470.4}$$
$$y_R = 1.11 \text{ m}$$

Curved Surface, Summary



The net force is

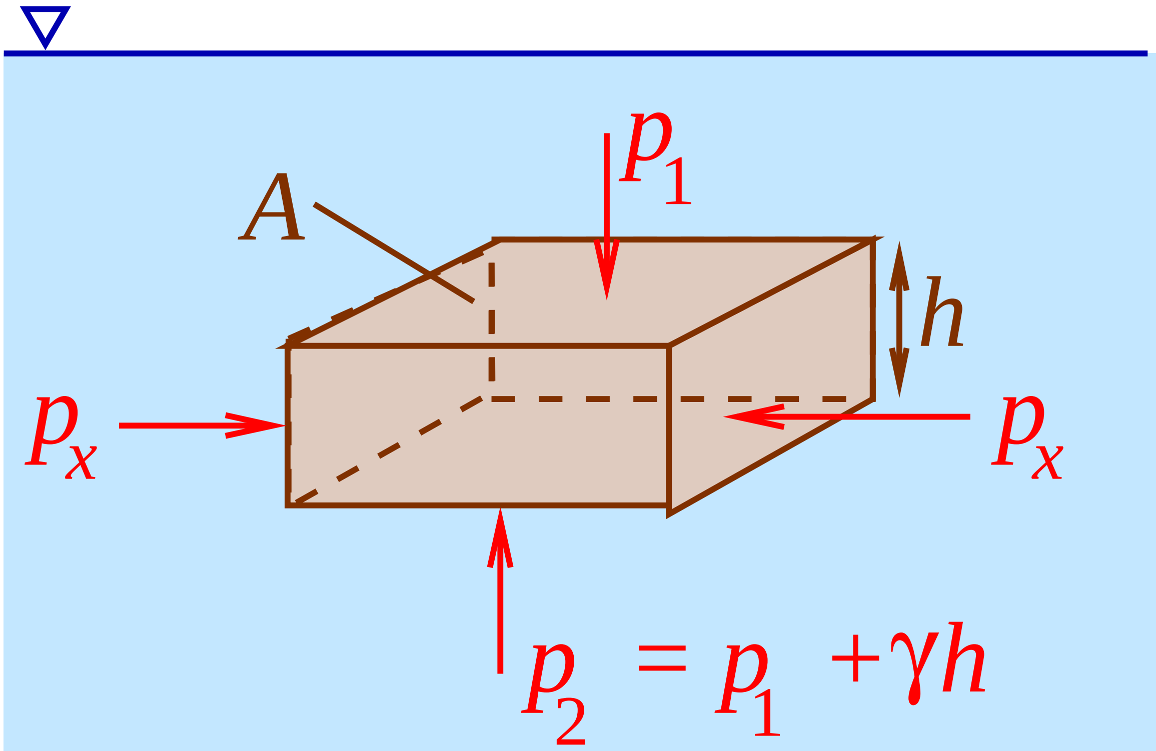
$$F_R = \sqrt{559.9^2 + 470.4^2} = 731.3 \text{ kN}$$

Static balance and Action and Reaction can be applied

$$F_{\text{PressurePlusWeightOnWedge}} + F_{\text{TankOnWedge}} = 0$$
$$F_{\text{TankOnWedge}} + F_{\text{WedgeOnTank}} = 0$$

Buoyancy: Archimedes principle

When a body is wholly or partially immersed in a fluid there is an upward buoyancy force equal to the weight force of the fluid displaced by the body.



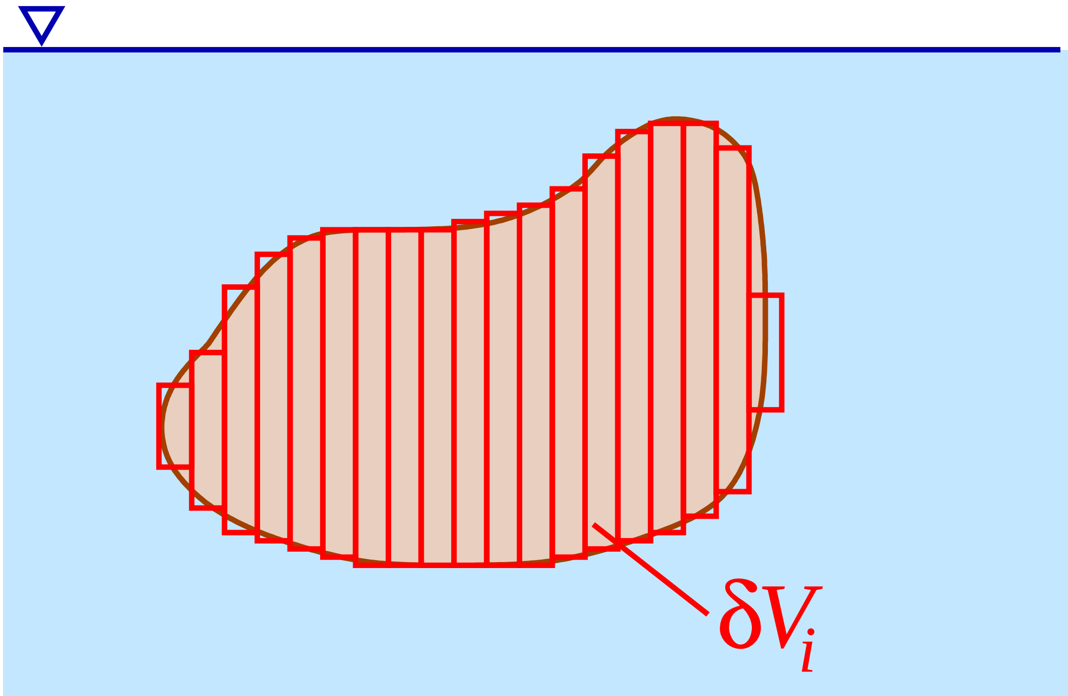
Consider pressure forces on a rectangular slab

$$
\begin{aligned}
F_{\text{net:pressure}} &= p_2 A - p_1 A \\
F_{\text{net:pressure}} &= (p_1 + \gamma h) A - p_1 A \\
F_B &= \gamma h A = \gamma V
\end{aligned}
$$

The buoyancy force arises as a result of higher pressure on the bottom surface.

Archimedes principle: Proof

Any body can be decomposed into a number of very small slabs. The buoyancy force on each slab is just $\delta F_i = \gamma \delta V_i$.



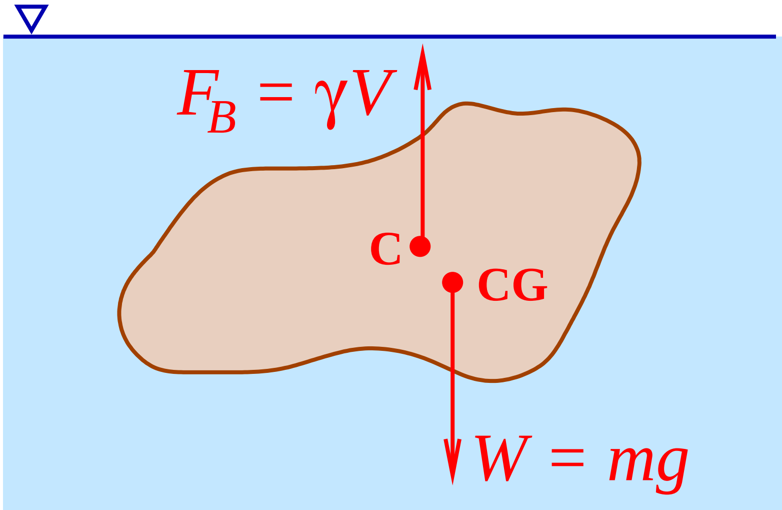
Therefore the proof for a rectangular slab can be generalized to a body of arbitrary shape.

$$F_B = \sum_i \delta F_i = \gamma V.$$

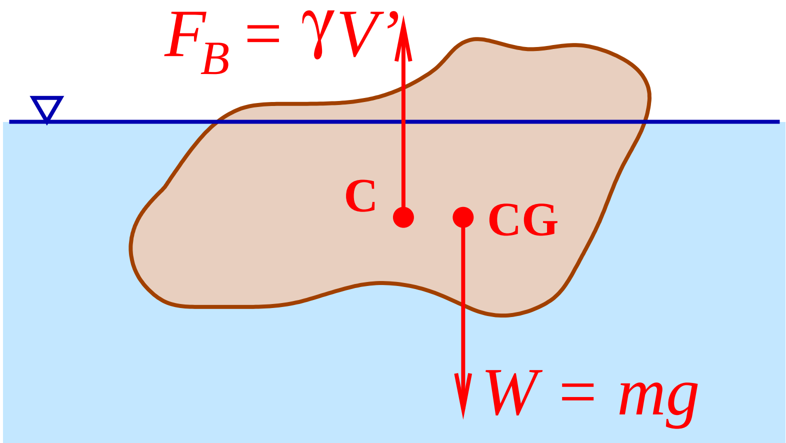
Buoyancy force does not depend on the density of the submerged object. The buoyancy force only depends on the density of the fluid and the volume(shape) of the submerged object.

Archimedes principle

The Buoyancy force of a submerged body passes through its centroid. Called the center of buoyancy.



The buoyancy force for a partially submerged object passes through the centroid of the displaced volume, V' .



The weight force passes through the center of gravity and does not always pass through the centroid.